AP Calculus BC – McGlone
Section 10.6 – Arc Length in Polar (Anton new; FDWK)

Find the arc length of the polar curve. Assume the entire curve unless noted otherwise.

1. $r = a$

2. $r = 2a\cos\theta$

3. $r = a - a\cos\theta$

4. $r = \sin^2\left(\frac{\theta}{2}\right)$, from $\theta = 0$ to $\theta = \pi$

5. $r = e^{3\theta}$, from $\theta = 0$ to $\theta = 2$

6. $r = \sin^3\left(\frac{\theta}{3}\right)$, from $\theta = 0$ to $\theta = \frac{\pi}{2}$

7. $r = \theta^2,\ 0 \le \theta \le \sqrt{5}$

8. $r = \frac{e^\theta}{\sqrt{2}},\ 0 \le \theta \le \pi$

9. $r = \sqrt{1 + \sin 2\theta}\,,\ 0 \le \theta \le \pi\sqrt{2}$

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Find the arc length of the polar curve. Assume the entire curve unless noted otherwise.

1. $r = a$

$\frac{dr}{d\theta} = 0$

$\int_0^{2\pi} \sqrt{a^2 + 0^2}\, d\theta$

$\int_0^{2\pi} a\, d\theta$

$a\theta\Big]_0^{2\pi} = \boxed{2\pi a}$

2. $r = 2a\cos\theta$

$\frac{dr}{d\theta} = -2a\sin\theta$

$\int_0^{\pi} \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta}\, d\theta$

$\int_0^{\pi} 2a\, d\theta$

$2a\theta\Big]_0^{\pi} = \boxed{2a\pi}$

3. $r = a - a\cos\theta$

$\frac{dr}{d\theta} = a\sin\theta$

$\int_0^{2\pi} \sqrt{(a - a\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$

$a\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$

$a\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$

$\rightarrow a\int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$

$2 \cdot 2a\int_0^{\pi} \sin\frac{\theta}{2}\, d\theta$

$-8a\cos\frac{\theta}{2}\Big]_0^{\pi} = -8a(0 - 1) = \boxed{8a}$

4. $r = \sin^2\left(\frac{\theta}{2}\right)$, from $\theta = 0$ to $\theta = \pi$

$\frac{dr}{d\theta} = \cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\int_0^{\pi} \sqrt{\sin^4\frac{\theta}{2} + \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\, d\theta$

$\int_0^{\pi} \sin\frac{\theta}{2}\, d\theta = -2\cos\frac{\theta}{2}\Big]_0^{\pi} = -2\left(\cos\frac{\pi}{2} - \cos 0\right) = \boxed{2}$

5. $r = e^{3\theta}$, from $\theta = 0$ to $\theta = 2$

$\frac{dr}{d\theta} = 3e^{3\theta}$

$\int_0^{2} \sqrt{e^{6\theta} + 9e^{6\theta}}\, d\theta$

$\sqrt{10}\int_0^{2} e^{3\theta}\, d\theta = \frac{\sqrt{10}}{3} e^{3\theta}\Big]_0^{2} = \boxed{\frac{\sqrt{10}}{3}\left(e^{6} - 1\right)}$

6. $r = \sin^3\left(\frac{\theta}{3}\right)$, from $\theta = 0$ to $\theta = \frac{\pi}{2}$

$\frac{dr}{d\theta} = \cancel{3}\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin^6\frac{\theta}{3} + \sin^4\frac{\theta}{2}\cos^2\frac{\theta}{2}}\, d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^2\frac{\theta}{2}\, d\theta$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} 1 - \cos\theta\, d\theta$$

$$\frac{1}{2}(\theta - \sin\theta)\Big]_0^{\pi/2} = \boxed{\frac{1}{2}\left(\frac{\pi}{2} - 1\right)}$$

7. $r = \theta^2,\ 0 \le \theta \le \sqrt{5}$

$\frac{dr}{d\theta} = 2\theta$

$$\int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

$$\int_0^{\sqrt{5}} \sqrt{\theta^2(\theta^2+4)}\, d\theta$$

$$\int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\, d\theta$$

$u = \theta^2+4$

$du = 2\theta\, d\theta$

$$\frac{1}{2}\int_4^9 u^{1/2}\, du$$

$$\frac{1}{3} u^{3/2}\Big]_4^9$$

$$\frac{1}{3}\left(9^{3/2} - 4^{3/2}\right) = \boxed{19/3}$$

8. $r = \frac{e^\theta}{\sqrt{2}},\ 0 \le \theta \le \pi$

$\frac{dr}{d\theta} = \frac{e^\theta}{\sqrt{2}}$

$$\int_0^{\pi} \sqrt{\frac{e^{2\theta}}{2} + \frac{e^{2\theta}}{2}}\, d\theta$$

$$\int_0^{\pi} e^\theta\, d\theta$$

$$e^\theta\Big]_0^{\pi} = \boxed{e^\pi - 1}$$

9. $r = \sqrt{1 + \sin 2\theta},\ 0 \le \theta \le \pi\sqrt{2}$

$r^2 = 1 + \sin 2\theta$

$\cancel{2}r\frac{dr}{d\theta} = \cancel{2}\frac{\cos 2\theta}{r}$

$$\int_0^{\pi\sqrt{2}} \sqrt{(1+\sin 2\theta) + \frac{\cos^2 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$\int_0^{\pi\sqrt{2}} \sqrt{\frac{(1+\sin 2\theta)^2 + \cos^2 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$\int_0^{\pi\sqrt{2}} \sqrt{\frac{2 + 2\sin 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$\int_0^{\pi\sqrt{2}} \sqrt{2}\, d\theta = \sqrt{2}\theta\Big]_0^{\pi\sqrt{2}} = \boxed{2\pi}$$